Motivic p-adic L-functions

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Introduction. The connexions between special values of L-functions and arithmetic is an ancient and mysterious theme in number theory, which can be traced through the work of Dirichlet, Kummer, Minkowski, Siegel, Tamagawa, Weil, Birch and Swinnerton-Dyer, Iwasawa, ... Recently, Bloch and Kato [1], using ideas which rely heavily on the work of Fontaine, have succeeded in formulating a very general version of the classical Tamagawa number conjecture for linear algebraic groups for arbitrary motives over the rational field Q, which seems to contain as special cases all earlier conjectures about these questions. Needless to say, only a very modest amount of progress has been made so far towards proving the Bloch-Kato conjecture for specific motives over Q (essentially, the only cases where it can be established at present are for the Tate motives, and certain motives arising from elliptic curves with complex multiplication). In all the cases where proofs are known, the conjecture is established for each prime p separately, and the deepest part of the argument involves ideas from Iwasawa theory. Specifically, one must use a version for the motive of the so called 'main conjecture' of Iwasawa theory, which has now been completely proven for the above motives (apart from the troublesome primes 2 and 3 in the case of elliptic curves with complex multiplication), thanks to the beautiful work of Mazur, Wiles, Thaine, Kolyvagin and Rubin (see the article by Rubin in this volume). It does at least make sense to try to formulate the 'main conjecture' for arbitrary motives over Q, although one should have no illusions about the difficulty of proving it. The formulation of this 'main conjecture' involves, on the one hand, p-adic Iwasawa modules which are built out of the representations of the Galois group of Q given by the p-adic realisations of the motive (see the article by Greenberg in this volume for a discussion of the case when p is ordinary), and on the other hand, p-adic avatars of the complex L-function of the motive, which are built out of the critical special values of the complex L-function. The aim of the present article is to give a detailed conjectural description of these motivic p-adic L-functions, at least for primes p for which the motive has good ordinary reduction. Nearly everything which is contained in this paper is already given in the earlier articles by B. Perrin-Riou and myself ([3], [4], [10]). However, the assertions made about holomorphy in these earlier papers were too strong, and I have, I hope, corrected these here, as well as giving somewhat fuller versions of some of the crucial arguments about modifications of the Euler factors at both finite and infinite primes.

1. Notation and normalization. Let Q denote the field of rational numbers and C (resp. R) the field of complex numbers (resp. real numbers). Throughout, p will signify an arbitrary prime number (we do not exclude p=2), and we write Z_p, Q_p, C_p for the ring of p-adic integers, the field of p-adic numbers, and the completion of an algebraic closure of the field of p-adic numbers. Let U denote the group of units of Z_p. Let A denote the algebraic closure of Q in C. We fix, once and for all, an embedding

$$j: A \rightarrow C_p \qquad (1)$$

which we will often not make explicit in our formulae. Let Q^{ab} be the maximal abelian extension of Q in A. If K/F is a Galois extension of fields, we write G(K/F) for the Galois group of K over F. For brevity, we put

$$G = G(A/Q) \ , \quad G^{ab} = G(Q^{ab}/Q) \ .$$

We use the embedding (1) to identify complex and p-adic characters of finite order of G^{ab}. For each integer $m \geq 1$, let μ_m denote the group of m-th roots of unity in A. Let Ξ be the group of all p-power roots of unity, and put

$$P = Q(\Xi) \ , \quad H = Q(\Xi)^+ \ , \quad J = G(H/Q) \ . \qquad (2)$$

(here the + denotes the maximal real subfield). We write

$$\psi : G(P/Q) \rightarrow U \qquad (3)$$

for the isomorphism given by the action of this Galois group on W, i.e. $\zeta^\sigma = \zeta^{\psi(\sigma)}$ for all ζ in Ξ and σ in Q. We also put

$$X = \mathrm{Hom}_{\mathrm{cont}}(J, C_p^*) \ . \qquad (4)$$

As far as the sign of the reciprocity law map is concerned, we must stress that we adopt throughout the geometric convention of [5], rather than the more classical arithmetic convention. Specifically, this convention is as follows. For each finite prime q, let Frob_q denote the arithmetic Frobenius, i.e. it operates on the algebraic closure of the field with q elements by sending x to x^q. Let C denote the idele class group of Q. Let x_q be any idele whose q -th component is a local parameter at q, and all of whose other components are equal to 1. Then we choose the sign of the reciprocity map

$$r : C \rightarrow G^{ab} \qquad (5)$$

such that $r(x_q)$ is an element of G^{ab} which acts on the algebraic closure of the residue field at q via the *inverse* of Frob_q. Let $\gamma : C \rightarrow C^*$ be a continuous homomorphism. The complex L-function of γ is then defined, as usual, by the Euler product

$$L(\gamma, s) = \prod (1 - \gamma(x_q)/q^s)^{-1} \ , \qquad (6)$$

where the product is taken over all finite primes q which are not ramified for γ, and x_q is as above. Similarly, if S is any finite set of primes of Q, we write $L_S(\gamma, s)$ for the function obtained by omitting from (6) all Euler factors at primes which lie in S. Now let $\phi : G^{ab} \rightarrow A^*$ be any character of finite order. We define its associated idele class character

$$\phi_R : C \rightarrow A^* \qquad (7)$$

via the formula $\phi_R = \phi \circ r$. Thus, if q is a finite prime which does not divide the conductor of ϕ, we have

$$\phi_R(x_q) = \phi(\mathrm{Frob}_q^{-1}) \ . \qquad (8)$$

This last formula explains our choice of the sign of the reciprocity map (4), because it shows that the complex L-function (6) attached to ϕ_R coincides with the motivic L-function attached to ϕ (see §4).

2. p-adic pseudo-measures. The aim of this section is to give a slight generalization of the notion of a p-adic pseudo-measure which is given in [13]. Let O be the ring of integers of some finite extension of Q_p, and let I be any profinite abelian group (in the rest of the paper, we take I = J). For simplicity, let X also denote the group of continuous homomorphisms from I to C_p^*.The O - Iwasawa algebra $\mathfrak{I}$ of I is defined to be the projective limit of the group rings O[I/H], where H runs over the open subgroups of I. It is a compact algebra, which contains O[I] as a dense sub-algebra. The elements of $\mathfrak{I}$ are called integral measures on I (with values in O). This terminology is justified because, if μ is in $\mathfrak{I}$, and f is any continuous function from I to C_p, we can define the integral

$$\int_I f \, d\mu$$

by passage to the limit from the case when f is locally constant. In this latter case, if H is an open subgroup of I such that f is locally constant modulo H, and if the image of μ in O[I/H] is equal to $\Sigma\mu(s)s$, then the value of the above integral is equal to $\Sigma\mu(s)f(s)$, where, in both sums, s runs over I/H. We shall need the following generalization of the notion of an integral measure on I, in order to take into account possible poles of our p-adic L-functions. Let $Q(\mathfrak{I})$ be the ring of quotients of $\mathfrak{I}$, i.e. the ring of all quotients α/β, where α and β belong to $\mathfrak{I}$ and β is not a divisor of 0. An element μ of $Q(\mathfrak{I})$ is said to be a measure if there exists a non-zero element d of O such that $d\mu$ belongs to $\mathfrak{I}$. We say that an element μ of $Q(\mathfrak{I})$ is a *pseudo-measure* if there exists a non-zero element d of O, a finite subset S of X, and non-negative integers $n(\xi)$ $(\xi \in S)$, such that, for all choices of elements $\sigma(\xi)$ in I for ξ running over S, we have

$$d \, \Pi_{(\xi \in S)} \, (\xi(\sigma(\xi)) - \sigma(\xi))^{n(\xi)} \, \mu \tag{9}$$

belongs to the Iwasawa algebra $\mathfrak{I}$. It is clear that the pseudo-measures form a subring of $Q(\mathfrak{I})$. Suppose now that μ is a pseudo-measure. Let ϕ be any element of X which is distinct from all ξ in S. For each ξ in S, choose $\sigma(\xi)$ in I such that $\phi(\sigma(\xi)) \neq \xi(\sigma(\xi))$. We then define the integral of ϕ against μ by the formula

$$\int_I \phi \, d\mu \;=\; d^{-1} \, \Pi_{(\xi \in S)} \, (\xi(\sigma(\xi)) - \phi(\sigma(\xi)))^{-n(\xi)} \int_I \phi \, d\lambda , \tag{10}$$

where λ denotes the integral measure (9). It is immediately verified that this definition is independent of all choices. Also, if λ given by (9) is an integral measure, we say that the pseudo-measure μ has a pole at each ξ in S of order $\leq n(\xi)$; the minimal value of $n(\xi)$ such that the expression (9) lies in $\mathfrak{I}$ is called the exact order of the pole of μ at ξ.

Finally, there is an important involution on the ring of pseudo-measures on I, which we denote by $\mu \to \mu^{\#}$. This involution is given on O [I] by the O -linear map which sends σ to σ^{-1} for all σ in I, and it extends by continuity to $\mathfrak{I}$. It plainly extends to Q(I), and preserves the subring of pseudo-measures.

3. The cyclotomic theory. This section will be devoted to a brief account of the p-adic analogue of the Riemann zeta function. Recalling that X is given by (4), we write X_{alg} for the subgroup of X consisting of all characters of the form

$$\xi = \psi^n \chi \qquad (n \in Z) , \tag{11}$$

where χ is any character of finite order of G(P/Q), and ψ given by (3) is the p-adic cyclotomic character. Let τ_∞ denote the element of G given by complex conjugation. We are assuming that (11) is a character of the Galois group J, and this is clearly equivalent to the assertion that

$$\chi(\tau_\infty) = (-1)^n \; . \tag{12}$$

The following is the basic existence theorem for the p-adic analogue of the Riemann zeta function. Fix an integer $m \leq 0$ and a character ϕ of finite order of G(P/Q), which satisfy

$$\phi(\tau_\infty) = (-1)^{m-1} \quad (13)$$

The reason for this condition will become apparent later (in the notation and terminology explained later, we want the motive Q(m) twisted by ϕ to have weight ≥ 0 and to be critical at $s = 0$). Let $\mathcal{O}$ be the ring of integers of the field obtained by adjoining the values of ϕ to $\mathbf{Q}_p$, and let $\mathfrak{I}$ be the $\mathcal{O}$ - Iwasawa algebra of J. We remark that formula (15) below shows that, in the following theorem, the right hand side of (14) belongs to the field A of algebraic numbers, and so can be viewed as lying in $\mathbf{C}_p$ via the embedding (1).

Theorem 1. There exists a unique pseudo-measure $\mu = \mu(m,\phi)$ on the Galois group J satisfying :- (i). For all σ in J, $(\psi^{1-m}\phi^{-1}(\sigma) - \sigma)\mu$ belongs to the Iwasawa algebra $\mathfrak{I}$; (ii). If ξ given by (11) is any element of X_{alg} such that $m+n \leq 0$, then

$$\int_J \xi \, d\mu = L_T(\varpi_R, n+m), \quad (14)$$

where $T = \{p\}$, and $\varpi = \phi\chi$. Moreover, μ has a pole of order 1 at $\psi^{1-m}\phi^{-1}$.

We sketch what is essentially Iwasawa's proof of the existence of μ. Put r=4 or r=p, according as p is even or odd, and put $r_k = rp^k$ for all $k \geq 0$. For each p-adic unit u, write $[u]_k$ for its class in the group of relatively prime classes of integers modulo r_k. The partial zeta function

$$\zeta(u, r_k ; s) = \Sigma\, w^{-s} \qquad (R(s) > 1) \quad ,$$

where the sum is over all positive integers in $[u]_k$, has an analytic continuation over the whole complex plane, apart from a simple pole at s=1. For each non-negative integer t, we have

$$\zeta(u, r_k ; -t) = - r_k^t B_{t+1}(\{u\}_k / r_k)/(t+1) \quad , \quad (15)$$

where $\{u\}_k$ denotes the unique representative in Z of $[u]_k$, which lies between 0 and r_k ; here $B_{t+1}(x)$ denotes the (t+1)-th Bernoulli polynomial, which is defined by the expansion

$$ye^{yx}/(e^y - 1) = \Sigma_{(h\geq 0)} B_h(x)\, y^h/h! \quad .$$

In particular, we have

$$B_1(x) = x - 1/2\,, \quad B_{t+1}(x) = x^{t+1} - (t+1)/2\, x^t + \ldots \quad . \quad (16)$$

For t fixed, let p^e denote the largest power of p occurring in the denominators of the coefficients of $B_{t+1}(x)/(t+1)$. One deduces immediately from (15) and (16) that, for all integers $k \geq 0$ and all p-adic units u, we have

$$\zeta(u, r_{k+e} ; -t) \equiv t\, u^{t+1}/((t+1)r_{k+e}) + u^t \zeta(u, r_{k+e} ; 0) \bmod r_k . \quad (17)$$

If v is also a p-adic unit, we define

$$\delta_t(u, v; r_k) = v^{t+1} \zeta(u, r_k ; -t) - \zeta(uv, r_k ; -t).$$

Then we claim that, for all integers $k \geq 0$, we have

$$\delta_t(u, v; r_k) \equiv (uv)^t\, \delta_0(u, v; r_k) \bmod r_k \quad . \quad (18)$$

Note that (17) immediately implies the weaker version of (18), in which the first two r_k's appearing in (18) are replaced by r_{k+e}. But it is easy to see that this weaker congruence implies (18), when it is combined with the additional identity

$$\Sigma\, \zeta(z, r_h ; s) = \zeta(u, r_k ; s) \quad , \quad (19)$$

where h is any integer $\geq k$, and z runs over any set of representatives in U of those classes modulo r_h which map to the class of u modulo r_k. Note that one obvious consequence of (18) is that $\delta_t(u, v; r_k)$ is integral at p for all $t \geq 0$, because this is plainly true for $t = 0$ from the explicit formula (16).

We can now construct the pseudo-measure μ. For each u in U, let $\sigma(u)$ denote the unique element of G(P/Q) such that $\psi(\sigma(u)) = u$, and let $\tau(u)$ denote the restriction of $\sigma(u)$ to N. Let P_k be the field obtained by adjoining the group of r_k -th roots of unity to Q, and let N_k be the maximal real subfield of P_k. We write $\sigma_k(u)$ (resp. $\tau_k(u)$) for the restriction of $\sigma(u)$ to P_k (resp. to N_k). Write V_k for any set of representatives in U of the group of relatively prime residue classes modulo r_k. We assume in what follows that k is so large that the conductor of ρ divides r_k. For each p-adic unit v, define the following element of the O - group ring of the Galois group of N_k over Q

$$\lambda_k(v) = \phi(\sigma(v))^{-1} \Sigma\, \delta_{-m}(u, v; r_k)\, \tau_k(u)^{-1}\, \phi(\sigma_k(u))^{-1} ,$$

where the sum is over all u in V_k. The identity (19) shows that, as k varies, the $\lambda_k(v)$ define an element $\lambda(v)$ of the Iwasawa algebra $\mathfrak{I}$. Put $\theta = \psi^{1-m} \phi^{-1}$. By virtue of (13), θ is a character of the Galois group J. If v is not of finite order in U, it is easy to see that $\theta(\tau(v)) - \tau(v)$ is not a zero divisor in $\mathfrak{I}$. For any such v, we define

$$\mu = \lambda(v).(\theta(\tau(v)) - \tau(v))^{-1}.$$

It is readily verified that μ is a pseudo-measure on J, which is independent of the choice of v of infinite order in U, and which satisfies assertion (i) of the above theorem. Assume now that k is so large that r_k is also divisible by the conductor of χ. To prove (ii), we note that, by definition, the integral of $\xi = \psi^n \chi$ against the measure $\lambda(v)$ is the p-adic limit as $k \to \infty$ of the expression

$$\phi(\sigma(v))^{-1} \Sigma_{(u)}\, \delta_{-m}(u, v; r_k)\, u^{-n}\, \varpi(\sigma(u))^{-1}.$$

Applying the congruence (18) for t = - n and t = - n - m, we deduce that this limit is equal to the limit as $k \to \infty$ of the expression

$$\phi(\sigma(v))^{-1}\, v^n\, \Sigma_{(u)}\, \delta_{-n-m}(u, v; r_k)\, \varpi(\sigma(u))^{-1} .$$

But, again using (19), we see that this last quantity has a value independent of k, which is given by

$$(\theta(\tau(v)) - \xi(\tau(v))\, L_T(\varpi_R, n+m) .$$

Assertion (ii) is now plain. We omit the proof of the final statement of the theorem, which is a well known consequence of the von-Staudt-Clausen theorem giving the exact power of p occuring in the denominators of the k-th Bernoulli numbers, where k runs over the positive even integers which are divisible by p-1.

4. Complex L-functions. Motives arise in nature as direct summands of the cohomology of a given dimension of a smooth projective algebraic variety defined over Q. However, we shall simply view motives in the naive sense, as being defined by a collection of realisations, satisfying certain axioms. Moreover, since we must consider the twists of our motives by arbitrary characters of finite order of G^{ab}, it is technically necessary to consider motives over Q, with coefficients in some finite extension K of Q. A detailed account of such motives and their realizations is given in §2 of [6], and we only briefly recall some of the key definitions here. Let $\Sigma(K)$ denote the set of embeddings of K in the complex field C. We identify the C-algebras $K \otimes C$ (unless indicated to the contrary, all tensor products will be understood to be taken over Q) and $C^{\Sigma(K)}$ via

$$K \otimes C \cong C^{\Sigma(K)} \;:\; u \otimes w \to (w.\, \sigma(u))_\sigma . \tag{20}$$

In addition, for each prime number l, we put

$$K_l = K \otimes Q_l = \Pi_{(\lambda \mid l)} K_\lambda ,$$

where λ runs over the primes of K dividing l, and K_λ denotes the completion of K at λ. By a homogeneous motive M over Q, with coefficients in K, of weight w(M) and dimension d(M), we mean a collection of Betti $H_B(M)$, de Rham $H_{DR}(M)$, and l-adic $H_l(M)$ (one for each prime l) realisations, which are, respectively, free modules over K, K, and K_l, all of the same rank d(M). Moreover, these realisations are endowed with the following additional structure :- (i). $H_B(M)$ admits an involution F_∞ ; (ii). The global Galois group G has a continuous action on $H_l(M)$ for each prime l, and there is an isomorphism

$$g_l : H_B(M) \otimes_K K_l \to H_l(M)$$

which transforms the involution F_∞ into the complex conjugation; (iii). There is a decreasing exhaustive filtration $\{F^k H_{DR}(M) : k \in Z\}$ on the de Rham realisation; (iv). There is a Hodge decomposition into C-vector spaces

$$H_B(M)\otimes C = \oplus H^{i,j}(M) \quad , \tag{21}$$

where i,j run over a finite set of indices satisfying $i+j=w(M)$, and where F_∞ maps $H^{i,j}(M)$ to $H^{j,i}(M)$; (v). There is a $G_\infty = G(C/R)$ - isomorphism of C - vector spaces (which also commutes with the action of K)

$$g_\infty : H_B(M)\otimes C \rightarrow H_{DR}(M)\otimes C \tag{22}$$

where complex conjugation acts on the space on the right via its action on C, and on the space on the left via F_∞ on $H_B(M)$ and via its natural action on C; (vi). Finally, for all $k \in Z$, we have

$$g_\infty(\oplus_{(i\geq k)} H^{i,j}(M)) = F^k H_{DR}(M) \quad . \tag{23}$$

The first basic example of such a motive M is the Tate motive $Q(m)$, for any m in Z, which is of weight -2m and dimension 1. Let $V_l(\mu)$ be the tensor product over Z_l with Q_l of the projective limit of the Galois modules μ_{l^n} of l^n - th roots of unity, and let $V_l(\mu)^{\otimes m}$ be the m-th tensor power of $V_l(\mu)$. Then the realisations of $M = Q(m)$ are given by

$$H_B(M) = K \ , \ H_{DR}(M) = K \ , \ H_l(M) = V_l(\mu)^{\otimes m} \otimes_{Q_l} K_l \ .$$

The involution F_∞ is $(-1)^m$, and the action of G is the natural one. The Hodge decomposition is specified by taking $H^{-m,-m} = K\otimes C$, and the k-th term in the filtration of the de Rham cohomology is either K or 0, according as $k \leq -m$ or $k > -m$. The isomorphism (22) is given by $g_\infty(1\otimes 1) = 1\otimes(2\pi i)^m$.

If M is any such motive, we can construct the following motives from M :- (i). The twists M(n) for any n in Z ; the realisations of M(n) are the tensor products of the corresponding realisations of M and Q(n); (ii). The dual motive M^ ; the realisations of M^ are the dual vector spaces of the realisations of M.

We briefly recall the standard definitions and conjectures for the complex L-function attached to such a motive M. Put

$$\Gamma_R(s) = \pi^{-s/2}\Gamma(s/2) \ , \ \Gamma_C(s) = 2(2\pi)^{-s}\Gamma(s) \ .$$

For simplicity, we assume that, when w(M) is even, F_∞ acts on $H^{k,k}$, where $k = w(M)/2$, via a scalar (this will be automatically implied by our assumption made later that M is critical at s=0). As is explained in §2 of [6], the fact that $H_B(M)\otimes C$ is a free $K\otimes C$ - module, when with the identification (20), yields a decomposition

$$H_B(M)\otimes C = \oplus H_B(\sigma, M), \text{ where } H_B(\sigma, M) = H_B(M)\otimes_{(K,\sigma)} C \ ;$$

here σ runs over Σ(K) and the tensor product on the right is taken by regarding C as a K-algebra via σ. Each $H_B(\sigma, M)$ admits a Hodge decomposition

$$H_B(\sigma, M) = \oplus H^{j,k}(\sigma, M) \ ,$$

and we let $h(j,k)$ = C -dimension of $H^{j,k}(\sigma, M)$. This notation is justified, since it is shown in [6] that these dimensions are independent of $\sigma \in \Sigma(K)$. The Euler factor at ∞, which is also shown in [6] to be independent of the choice of σ, is then defined by

$$L_\infty(M, s) = L_\infty(\sigma, M, s) = \prod_{(U)} L_\infty(U, s) \ ,$$

where U runs over the direct summands of $H_B(\sigma, M)$ of either the form (i) $U = H^{j,k}(\sigma)\oplus H^{k,j}(\sigma)$ with $j < k$, or (ii) $U = H^{k,k}(\sigma)$, (where we have abbreviated $H^{j,k}(\sigma,M)$ by $H^{j,k}(\sigma)$) and $L_\infty(U, s)$ is given explicitly by :- (a). In case (i), $L_\infty(U, s) = \prod_{(j<k)} \Gamma_C(s-j)^{h(j,k)}$; (b). In case (ii) when F_∞ acts on $H^{k,k}(\sigma)$ via $(-1)^k$, then $L_\infty(U,s) = \Gamma_R(s-k)^{h(k,k)}$; (c). In case (ii) when F_∞ acts on $H^{k,k}(\sigma)$ via $(-1)^{k+1}$,then $L_\infty(U, s) = \Gamma_R(s+1-k)^{h(k,k)}$. By contrast, the Euler factors at finite primes do depend on the choice of σ in Σ(K). If q is a finite prime, let I_q denote the inertial subgroup in G of some fixed prime of A lying above q. Then the Euler factor at q is given by

$$L_q(\sigma, M, s) = (\sigma Z_q)(M, q^{-s}),$$

where

$$Z_q(M, X) = \det(1 - \mathrm{Frob}_q^{-1}.X \mid H_\lambda(M)^{I_q})^{-1} ,$$

and where λ is any prime K not lying above q. We have imposed the standard hypothesis that $Z_q(M, X)$ is a rational function in X, with coefficients in K, which are independent of the choice of the prime λ. The complex L-function of M is then defined by the Euler product

$$\Lambda(\sigma, M, s) = \prod L_v(\sigma, M, s) ,$$

where v runs over all primes of Q, including $v = \infty$. We also write $L(\sigma, M, s)$ for this Euler product with the infinite Euler factor omitted. Note that we have

$$\Lambda(\sigma, M(n), s) = \Lambda(\sigma, M, s+n) \text{ for all } n \in \mathbf{Z} .$$

We assume that there exists a finite set of primes $S = S(M)$ such that (i) for each prime λ, and each q which is not in S and which does not lie below λ, the inertia group I_q operates trivially on $H_\lambda(M)$, and (ii) for q not in S, the reciprocal complex roots of $(\sigma Z_q)(M, X)^{-1}$ have absolute value equal to $q^{w(M)/2}$. Under additional hypotheses, one can then define the conductor of M and the global ε-factor $\varepsilon(\sigma, M, s)$ (see [14]). Here is the standard conjecture about the analytic continuation and functional equation of this L-function.

Conjecture A (Complex Version). $\Lambda(\sigma, M, s)$ has a meromorphic continuation over the whole complex plane to a function of order ≤ 1, and satisfies the functional equation

$$\Lambda(\sigma, M, s) = \varepsilon(\sigma, M, s)\Lambda(\sigma, M\hat{\ }(1), -s) \quad . \tag{24}$$

It is also conjectured that $\Lambda(\sigma, M, s)$ is entire if $w(M)$ is odd, and that the only possible pole which can occur, if $w(M)$ is even, is at $s = 1+w(M)/2$. In this latter case, the order of the pole is conjectured to be the K_λ- dimension of the subspace of $H_\lambda(M(w(M)/2))$ which is fixed by the global Galois group G, for any prime λ of K.

As is explained in [5] and [14], the global ε-factor $\varepsilon(\sigma, M, s)$ has a decomposition into local factors, which we shall see plays an important role in the non-archimedean theory. Let Θ denote the adele group of Q. Fix, once and for all, the Haar measure $dx = \prod dx_v$ on Θ, where dx_∞ is the usual Haar measure on R, and, for each prime q, dx_q is the Haar measure on Q_q which gives Z_q volume 1. To define the local ε-factors, we must choose a complex character of the adele class group Θ/Q, and there are inescapably two natural choices. For the rest of this paper, ρ will denote i or -i, where i has its usual meaning as a complex number. Let η_ρ denote the character of Θ/Q with components $\eta_{\rho,\infty}(x) = \exp(2\pi\rho x)$, and, for each finite prime q, $\eta_{\rho,q}(x) = \exp(-2\pi\rho x)$, where we have identified Q_q/Z_q with the q-primary subgroup of Q/Z. For each place v of Q,, let $\varepsilon_v(\sigma, M, \rho, s)$ denote Deligne's local ε-factor for the relative to the various choices just described (we have suppressed the the fixed measure dx_v in the notation, and we simply write ρ instead of the additive character η_ρ). Then we have

$$\varepsilon(\sigma, M, s) = \prod_v \varepsilon_v(\sigma, M, \rho, s) , \tag{25}$$

where the product is taken over all primes v of Q, including $v=\infty$. Note also that we have the fundamental relation

$$\varepsilon_v(\sigma, M, \rho, s)\, \varepsilon_v(\sigma, M\hat{\ }(1), -\rho, -s) = 1 \quad . \tag{26}$$

It is fundamental for the non-archimedean theory that we also consider the twists of our motive M by characters of finite order of G^{ab}, and we now briefly recall the definition of these twists. Let $\phi: G^{ab} \to A^*$ be a character of finite order, and assume that the values of ϕ lie in K. Following [6], §6, we can attach to ϕ a motive $[\phi]$ of dimension 1 and weight 0 over Q, with coefficients in K. Let $V(\phi)$ be the vector space of dimension 1 over K, on which G acts via ϕ. We then define $H_B(\phi)$ to be the underlying space of $V(\phi)$, with the action of F_∞ given by $\phi(\tau_\infty)$, where τ_∞ is complex conjugation. The de Rham realisation is given by $H_{DR}(\phi) = (V(\phi)\otimes A)^G$, where the global Galois group G acts both on $V(\phi)$ via ϕ, and on A in the natural fashion (we endow the de Rham realisation with the trivial filtration for which F^k is 0 for $k > 0$, and the whole space for $k \le 0$). The comparison isomorphism

$$g_{\phi,\infty}: H_B(\phi)\otimes C \to H_{DR}(\phi)\otimes C \tag{27}$$

is obtained by noting that $H_{DR}(\phi)$ provides a Q - structure for $H_B(\phi)\otimes A$, and then extending scalars from A to C. For each finite prime λ of K, we take the λ-adic realisation to be $H_\lambda(\phi)$ to be a vector space of dimension 1 over the completion K_λ of K at λ, on which G acts via ϕ. For each embedding σ: K $\to$ C, we can apply the above motivic recipe for attaching a complex L-function L(σ, ϕ, s) to the motive [ϕ], and, in view of (8), we see that L(σ, ϕ, s) coincides with the L-function $L((\phi^\sigma)_R$, s) defined by (6) - indeed, our sign of the reciprocity map was chosen to assure this. Now let M be a motive over Q , with coefficients in K, as above. The twist M(ϕ) is then defined to be the motive over Q , with coefficients in K, whose realisations are the tensor products over K of the realisations of M with the corresponding realisations of [ϕ].

5. Critical points and the period conjecture. Our goal in this section is to give a modified version of Deligne's period conjecture of [6], which seems essential for problems of p-adic interpolation. We shall be concerned with the following question. Let M be a fixed motive over Q , with coefficients in some finite extension K of Q , and consider twists of M of the form

$$W = M(n)(\phi) \ , \text{ with } \phi(\tau_\infty) = (-1)^n \ , \tag{28}$$

where n ranges over Z , and ϕ over the characters of finite order of G^{ab} with values in K. How does the Deligne period $c^+(W)$ vary with n and ϕ? It turns out that the naive answer to this question is not precise enough for problems of p-adic interpolation, and our aim will be to use the properties of the complex L-function to give a finer answer, at least when both M and W are critical at s = 0.

We begin by briefly explaining the naive answer to the above question, which does not depend on any assumptions about M or W being critical at s = 0. In fact, the techniques of [6] reduce this to a problem of linear algebra (see [6] for the background material, which we do not repeat in detail here). We suppose always that K contains the values of ϕ. We assume that F_∞ acts on $H^{k,k}(M)$ by a scalar. In §2 of [6], Deligne attaches to W a period $c^+(W)$ in $(K\otimes C)^*$, which is well defined up to multiplication by an element of K^*. Let ρ denote a choice of either +i or -i in C. Let f(ϕ) denote the conductor of ϕ , so that ϕ factors through the Galois group, which we denote by $\Delta(\phi)$, of the

field generated over Q by the group of f(ϕ)-th roots of unity. Following §6 of [6], we define the element $\delta_\rho(\phi) \in (K\otimes C)^*$ by

$$\delta_\rho(\phi) = \Sigma_{(\tau\in\Delta(\phi))}\, \phi^{-1}(\tau)\otimes(\exp(-2\pi\rho/f(\phi)))^\tau \quad . \tag{29}$$

If α = + or -, let $H_B(M)^\alpha$ denote the subspace of $H_B(M)$ on which F_∞ acts via the sign α, and let $d^\alpha(M)$ denote its K- dimension.

Lemma 2. Let W be the motive given by (28). Then, up to multiplication by an element of K^*, $c^+(W)$ coincides with

$$c^+(M)((2\pi i)^n\, \delta_\rho(\phi))^{d^+(M)} . \tag{30}$$

Proof. Let T = M(n), and put $\varepsilon = \phi(\tau_\infty)$. Then (28) implies (see [6], p. 329) that

$$c^\varepsilon(T) = (2\pi i)^{nd^+(M)}\, c^+(M) \ . \tag{31}$$

Now W = T(ϕ), and (28) gives immediately

$$H_B(W)^+ = H_B(T)^\varepsilon \otimes_K H_B(\phi) \ , \ H_{DR}(W)^+ = H_{DR}(T)^\varepsilon \otimes_K H_{DR}(\phi) \ .$$

By definition, $c^+(W)$ is the determinant of the comparison isomorphism

$$g_{R,\infty}{}^+ : H_B(W)^+ \otimes C \to H_{DR}(W)^+ \otimes C \ ,$$

computed relative to K-bases of the two sides (which each have cardinality equal to $d^+(M)$). Now a K-basis of the left hand side is given by $\{\alpha_i\otimes1\otimes1\}$, where $\{\alpha_i\}$ is a K-basis of $H_B(T)^\varepsilon$. Similarly, a K-basis of the right hand side is given by $\{\beta_i\otimes g\otimes1\}$, where g is any non-zero element of $H_{DR}(\phi)$ and $\{\beta_i\}$ is a K-basis of $H_{DR}(T)^\varepsilon$. Thus $c^+(W)$ coincides, up to multiplication by an element of K^*, with coincides with $c^\varepsilon(T)g^{-d^+(M)}$. The assertion of the lemma now follows from (31), and the fact that, as remarked in §6 of [6], we can take $g=\delta_{-\rho}(\phi^{-1})/f(\phi)$, whence $g^{-1} = \delta_\rho(\phi)$.

Recall that an integer s = n is said to be *critical* for M if both the infinite Euler factors $L_\infty(\sigma,M, s)$ and $L_\infty(\sigma,M$^(1), -s) are holomorphic at s = n. The following lemma (due to Bloch, Deligne, Scholl, ...) gives several useful equivalent forms of this definition. As before, we write $h(j,k)$ = C-dimension of $H^{j,k}(\sigma) = H^{j,k}(\sigma, M)$ (see (24) and (25)), where $\sigma \in \Sigma(K)$. We recall that both the infinite Euler factors, and these dimensions, are independent of the choice of σ.

Lemma 3. The following three assertions are equivalent for M:- (i). M is critical at $s = 0$; (ii). If $j < k$ and $h(j,k) \neq 0$, then $j < 0$ and $k \geq 0$, and, in addition, if $h(k,k) \neq 0$, then F_∞ acts on $H^{k,k}(\sigma)$ by +1 if $k < 0$ and by -1 if $k \geq 0$; (iii). The map

$$h_\infty : H_B(M)^+ \otimes \mathbf{R} \rightarrow (H_{DR}(M)/F^0H_{DR}(M))\otimes\mathbf{R} \quad , \qquad (32)$$

induced from (22) is an isomorphism.

Proof. The equivalence of (i) and (ii) follows from the explicit formulae for the infinite Euler factors given above, and we do not give the details. Assume now that (ii) is valid. It follows that

$$d^+(M) = \Sigma_{(j<0)}\ h(j,k) \quad , \quad d^-(M) = \Sigma_{(j\geq 0)}\ h(j,k) \quad . \qquad (33)$$

It follows from (23) and these formulae that the two sides of (32) have the same $\mathbf{R}$ - dimension, and so it suffices to prove (32) is injective. Again by (23), h_∞ will certainly be injective if

$$(H_B(M)^+\otimes\mathbf{C}) \cap (\oplus_{(\sigma)}\oplus_{(j\geq 0)} H^{j,k}(\sigma)) = 0 \ . \qquad (34)$$

Let a $\mathbf{C}$ - basis of $H^{j,k}(\sigma)$ be given by $\{e_i(\sigma,j,k) : i = 1,...,h(j,k)\}$. Then a $\mathbf{C}$ - basis of $H_B(W)^+\otimes\mathbf{C}$ is given by the set

$$\{e_i(\sigma,j,k) + F_\infty(e_i(\sigma,j,k)) : j < 0 \leq k \ , i = 1,...,h_i(j,k),\ \sigma\in\Sigma(K)\},$$

together with the set $\{e_i(\sigma,k,k) : i = 1,...,h_i(k,k),\ \sigma\in\Sigma(K)\}$ if $k = w(W)/2$ is < 0. Hence any non-zero element of $H_B(W)^+\otimes\ \mathbf{C}$ will have a non-zero projection on at least one of the subspaces $H^{j,k}(\sigma)$ with $j < 0$. This proves (34), and so also (iii). Conversely, assume (iii) holds. The equality of dimensions on the two sides of (32) shows that (33) is then valid. But, if $j < k$, then the space $H^{j,k} \oplus H^{k,j}$ contributes $h(j,k)$ to both $d^+(M)$ and $d^-(M)$, and so it follows from (33) that we must have $j < 0 \leq k$. If $H^{k,k} \neq 0$, we also conclude from (33) that F_∞ acts on $H^{k,k}$ by +1 if $k < 0$, and by -1 if $k \geq 0$. This establishes (ii), and completes the proof of the lemma.

Fix an embedding σ in $\Sigma(K)$. If v is any place of $\mathbf{Q}$, we define

$$R_v(\sigma, M, \rho, s) = L_v(\sigma, M, s)/(\varepsilon_v(\sigma, M, \rho, s)L_v(\sigma,M\hat{}(1), - s)). \qquad (35)$$

As is already noted in [6] when v is non-archimedean (see Remark 5.2.1, p. 329), this ratio tends to be better behaved that the individual factors defining it. We shall exploit this fact in what follows. Clearly, we have

$$R_v(\sigma, M, \rho, s) = R_v(\sigma, M\hat{}(1), - \rho, - s)^{-1}.$$

It is therefore natural to ask whether one can define *canonical* new factors $E_v(\sigma, M, \rho, s)$ such that

$$R_v(\sigma, M, \rho, s) = E_v(\sigma, M, \rho, s)/E_v(\sigma, M\hat{}(1), - \rho, - s) \qquad (36)$$

(of course, this last equation cannot characterize the factors $E_v(\sigma, M, \rho, s)$). In fact this is the case, as we shall subsequently explain. Note one immediate consequence of such a construction. Let S be any finite set of primes of $\mathbf{Q}$. Define the modified L-function

$$\Lambda_{(S)}(\sigma, M, \rho, s) = \Pi_{(v\in S)}\, E_v(\sigma, M, \rho, s)\ .\ \Pi\, L_v(\sigma, M, \rho, s),$$

where the latter product is over primes v not in S. Then we have the following modified form of the functional equation (24)

$$\Lambda_{(S)}(\sigma, M, \rho, s) = (\Pi_{(v\notin S)}\, \varepsilon_v(\sigma, M, \rho, s))\ \Lambda_{(S)}(\sigma, M\hat{}(1), - \rho, - s). \qquad (37)$$

We now give the definition of the E_v - factors when $v = \infty$. For s in C, put

$$\rho^{-s} = \exp(-\rho\pi s/2),\ \Gamma_{\mathbf{C},\rho}(s) = \rho^{-s}\,\Gamma_{\mathbf{C}}(s)\,.$$

We also recall that the Euler factor at ∞ is independent of the choice of the embedding σ. Similarly, the ε - factor at ∞ is independent of the choice of σ (see, for example, the explicit formulae given on p. 329 of [6]), so that we may drop the σ from our notation in this case. We then define

$$E_\infty(M, \rho, s) = \Pi\, E_\infty(U, \rho, s), \qquad (38)$$

where U runs over the direct summands of the Hodge decomposition, and $E_\infty(U, \rho, s)$ is given explicitly by :-

(a) If $U = H^{j,k}(\sigma)\oplus H^{k,j}(\sigma)$ with $j<k$, then $E_\infty(U, \rho, s) = \Gamma_{C,\rho}(s - j)^{h(j,k)}$;

(b) If $U = H^{k,k}(\sigma)$ with $k\geq 0$, then $E_\infty(U, \rho, s) = 1$;

(c) If $U = H^{k,k}(\sigma)$ with $k<0$, then $E_\infty(U, \rho, s) = R_\infty(U, \rho , s)$.

From the table of values of the $\varepsilon_\infty(U, \rho, s)$ given on p. 329 of [6], we deduce easily that (36) is valid in case (a), and it is plainly true in cases (b) and (c).

If u and v are complex numbers, we write $u \sim v$ if there exists y in Q^* such that $u = yv$. The integer $r(M)$ defined by

$$r(M) = \Sigma_{(j<0)}\ jh(j,k)$$

plays an important role in what follows, thanks to the next crucial lemma.

Lemma 4. Assume that M is critical at $s = 0$. Then

$$E_\infty(M, \rho, 0) \sim (2\pi\rho)^{r(M)} \ , \tag{39}$$

where the rational number implicit in the $\sim$ is independent of the choice of ρ.

Corollary. Assume that M is critical at $s = 0$, and that W given by (28) is also critical at $s = 0$. Then

$$E_\infty(W, \rho, 0) \sim E_\infty(M, \rho, 0)(2\pi\rho)^{-nd^+(M)} \ . \tag{40}$$

To deduce the corollary, we first note that $h_M(j,k) = h_W(j-n, k-n)$, and that $d^+(M) = d^+(W)$, because of our hypothesis that $\phi(\tau_\infty) = (-1)^n$. As M is critical at $s = 0$, Lemma 3 shows that $d^+(M)$ is given by (33). On the other hand, since W is critical at $s = 0$, (ii) of Lemma 3 shows that $j-n < 0$ if and only if $j < 0$. Hence $r(W) = r(M) - nd^+(M)$, as required.

We now turn to the proof of Lemma 4. We prove the lemma by considering the three cases (a), (b), (c) for U above, and verifying (39) for each U, where $r(U) = jh(j,k)$ in case (a), $r(U) = 0$ in case (b), and $r(U) = kh(k,k)$ in case (c). This suffices since clearly $r(M) = \Sigma_{(U)}\ r(U)$. If s is in Z , we shall make use of the following classical properties of the Γ-function (see [6], p. 330) :-

$\Gamma_C(s) \sim (2\pi)^{-s}$ $(s>0)$, $\Gamma_R(s) \sim (2\pi)^{(1-s)/2}$ (s odd),
$\Gamma_R(s) \sim (2\pi)^{-s/2}$ (s>0 and even).

Suppose we are in case (a). Then

$$E_\infty(U, \rho, 0) = (\rho^j\,\Gamma_C(-j))^{h(j,k)} \sim (2\pi\rho)^{r(U)} \ ,$$

as required. In case (b), (39) is plainly valid. Suppose finally that we are in case (c) (the one delicate case). Put $h = h(k,k)$. There are two possibilities, according as k is even or odd. (i). Assume k is even, so that F_∞ acts on U by $(-1)^k$. Then $\varepsilon_\infty(U, \rho, 0) = 1$ by the table on p. 329 of [6], and we have

$$L_\infty(U, 0) = \Gamma_R(-k)^h \sim (2\pi)^{r(U)/2} \ , \ L_\infty(U\hat{\ }(1), 0) = \Gamma_R(k+1)^h \sim (2\pi)^{r(U)/2} \ ,$$

whence (39) holds. (ii). Assume k is odd, so that F_∞ acts on U by $(-1)^{k+1}$. Then $\varepsilon_\infty(U, \rho, 0) = \rho^{kh}$ by the table on p. 329 of [6], and we have

$$L_\infty(U,0) = \Gamma_R(1-k)^h \sim (2\pi)^{(k-1)h/2},\ L_\infty(U\hat{\ }(1),0) = \Gamma_R(j+2)^h \sim (2\pi)^{-(k+1)h/2},$$

whence (39) is again plain. The reader should also note that the unknown non-zero rational number implicit in (39) is independent of the choice of ρ.This completes the proof of Lemma 4.

We can now give the modified form of the period conjecture. Recall that we identify $K\otimes C$ with $C^{\Sigma(K)}$ via the isomorphism (20). Let $c^+(M) \in (K\otimes C)^*$ be the Deligne period of M as defined in §2 of [6], so that we can view $c^+(M)$ under the isomorphism (20) as a vector

$$c^+(M) = (c^+(\sigma, M))_{(\sigma \in \Sigma(K))} \ , \tag{41}$$

whose components are well defined up to multiplication by a system of numbers $\sigma(\alpha)$ $(\sigma \in \Sigma(K))$, for any α in K^*. For each choice of $\rho = i$ or $-i$, we now put

$$\Omega_\rho(M) = (\Omega_\rho(\sigma, M)) = c^+(M)(2\pi\rho)^{r(M)} . \tag{42}$$

If ϕ is a character of finite order of G^{ab} with values in K, we also recall that $\delta_\rho(\phi)$ is given by (29), and its image under the isomorphism (20) is $\delta_\rho(\phi) = (\delta_\rho(\sigma, \phi))$, where

$$\delta_\rho(\sigma, \phi) = \Sigma_{(\tau \in \Delta(\phi))} (\phi^\sigma(\tau))^{-1}(\exp(-2\pi\rho/f(\phi)))^\tau . \tag{43}$$

Lemma 5. Assume that M is critical at $s = 0$, and that W given by (28) is also critical at $s = 0$. Then, for each $\sigma \in \Sigma(K)$, the quantity

$$\Lambda_{(\infty)}(\sigma, W, \rho)\, \Omega_\rho(\sigma, M)^{-1}\, \delta_\rho(\sigma, \phi)^{-d^+(M)} \tag{44}$$

does not depend on the choice of $\rho = i$ or $-i$.

Proof. It is plain that

$$\delta_{-\rho}(\sigma, \phi) = \phi^\sigma(\tau_\infty)\, \delta_\rho(\sigma, \phi). \tag{45}$$

The assertion of the lemma now follows from (28), Lemma 4, and the fact noted earlier that $r(W) = r(M) - nd^+(M)$.

Period Conjecture. Assume that M is critical at $s = 0$, and that W given by (28) is also critical at $s = 0$. Then there exists $\alpha \in K$ such that the expression (44) is of the form $\sigma(\alpha)$ for all $\sigma \in \Sigma(K)$.

Indeed, by Lemmas 3 and 4, we see that this conjecture is equivalent to Conjecture 2.8 of [6], applied to the motive W.

6. Modification of the Euler factor at p. We again let M be any motive over Q with coefficients in K. For this section, we drop the assumption that M is critical at $s = 0$, as it will not be needed. Let p be any prime number - the only restriction placed on p is given by Hypothesis I(p) below. Our aim in this section is to define a modification of the Euler factor at p, which is analogous to that already

given for the Euler factor at ∞. Throughout, σ will denote any element of $\Sigma(K)$, and ρ will again denote i or -i.

Let G_p denote the absolute Galois group of Q_p, and let I_p (resp. W_p) be the subgroup of G_p given by the inertial subgroup (resp. the Weil group). We fix an element Φ of G_p, whose image in G_p/I_p is the geometric Frobenius (i.e. the inverse of Frob_p). For each $s \in C$, let

$$\omega_s : W_p \to C^*$$

be the homomorphism which is trivial on I_p, and which satisfies $\omega_s(\Phi) = p^{-s}$. We also fix a prime number $l \neq p$, and a non-zero homomorphism $t_l : I_p \to Z_l$. Let λ denote a prime of K above l, and K_λ the completion at λ. Now write W_p' for the Weil - Deligne group of Q_p. Recall that the representations of W_p' are defined as follows (see [5], §8). Let V be a finite dimensional vector space over K_λ. Then a representation of W_p' in V is a pair $\Theta = (\gamma, N)$, where (i) $\gamma : W_p \to GL(V)$ is a homomorphism, whose kernel contains an open subgroup of I_p, and (ii) N is a nilpotent endomorphism of V such that

$$\gamma(\sigma)\, N\, \gamma(\sigma)^{-1} = \omega_1(\sigma)\, N \qquad \text{for all } \sigma \text{ in } W_p .$$

Given such a representation Θ, we can define the dual representation $\Theta^\wedge = (\gamma^\wedge, N^\wedge)$, where $\gamma^\wedge$ is the contragredient representation. Writing $I = I_p$ for brevity, we define

$$V_N = \mathrm{Ker}\,(N) , \quad Z_p(\Theta, X) = \det(1 - \gamma(\Phi)\, X \mid V_N^{\gamma(I)})^{-1} .$$

Let $\sigma : K_\lambda \to C$ denote a fixed extension of the embedding σ in $\Sigma(K)$. We then put $L_p(\sigma, \Theta, s) = (\sigma Z_p)(\Theta, p^{-s})$. Write $\varepsilon_p(\sigma, \gamma\otimes\omega_s, \rho)$ for Deligne's ε-factor attached to the representation $\gamma\otimes\omega_s$ of W_p on the complex vector space $V\otimes_{(K_\lambda, \sigma)}C$, and define

$$\varepsilon_p(\sigma, \Theta, \rho , s) = \varepsilon_p(\sigma, \gamma\otimes\omega_s , \rho) \det(-\Phi.\, p^{-s} \mid V^{\gamma(I)}/V_N^{\gamma(I)}) .$$

By analogy with (35), we can now define $R_p(\sigma, \Theta, \rho, s)$ by simply replacing M throughout by Θ in the formula (35). As pointed out in Remark 5.2.1 of [6], the expression $R_p(\sigma, \Theta, \rho, s)$ is particularly well behaved. In particular, it does not change if we replace the representation $\Theta = (\gamma, N)$ by $\Theta' = (\gamma, 0)$. Finally, we recall (see [5], §8.5) that, for each representation $\Theta = (\gamma, N)$ of W_p', one can define a new representation $\Theta^{ss} = (\gamma^{ss}, N)$ called the Φ-semisimplification of Θ, which has the property that γ^{ss} is a semisimple representation of the ordinary Weil group W_p. Again, $R_p(\sigma, \Theta, \rho, s)$ does not change if we replace Θ by Θ^{ss}.

Now let us return to the λ-adic representation of W_p given by its natural action on $H_\lambda(M)$. By Grothendieck's theorem, this λ-adic representation gives rise to a unique representation $\Theta = (\gamma, N)$ of the Weil - Deligne group W_p' (see [5], §8).

Lemma 6. There exists a representation $\Theta' = (\gamma', N')$ of the Weil-Deligne group in $H_\lambda(M)$, which satisfies :- (i). $N' = 0$; (ii). If we extend scalars from K_λ to $\mathbf{C}$ via the embedding σ, then γ' is a semisimple complex representation of W_p; and (iii). We have

$$R_p(\sigma, M, \rho, s) = R_p(\sigma, \Theta', \rho, s). \tag{46}$$

Proof. By the construction of the representation Θ via Grothendieck's theorem, we have that (46) is valid with Θ replaced by Θ'. On the other hand, it was remarked above that $R_p(\sigma, \Theta, \rho, s)$ does not change if we replace Θ by $\Theta_1 = (\gamma, 0)$, and subseqently Θ_1 by $\Theta' = \Theta_1^{ss}$. It is plain that this choice of Θ' satisfies the assertions of the lemma.

We can now define the factors $E_p(\sigma, M, \rho, s)$ satisfying (36). Let

$$\gamma : W_p \to GL(Y), \quad \text{where } Y = H_\lambda(M) \otimes_{(K_\lambda, \sigma)} \mathbf{C} \tag{47}$$

be the semisimple complex representation of the Weil group given by Lemma 6. Let $Y = \oplus U$ be the decomposition of Y into irreducible complex representations of W_p. For each such representation U, we can define the expression $R_p(\sigma, U, \rho, s)$ by the formula (35) with M replaced by U, and, in view of (46), we have

$$R_p(\sigma, M, \rho, s) = \Pi_{(U)} R_p(\sigma, U, \rho, s). \tag{48}$$

Each U occurring in this decomposition is irreducible, and hence is known (see [5] , §4.10) to be of the form $\xi_U \otimes \omega_{s(U)}$, where s(U) is some complex number, and ξ_U is a complex representation of the Weil group such that $\xi_U(W_p)$ is a finite group. Consequently, the inverse roots of the polynomial det $(1 - \Phi. X \mid U)$ (note that we do not take the subspace of U fixed by I, but rather the whole of U) are all of the form a root of unity times one fixed root. Thus, assuming that these inverse roots are algebraic numbers, and viewing them as lying inside $\mathbf{C}_p$ via the embedding (1), we can unambiguously define $ord_p(U)$ to be $ord_p(\alpha)$ for any inverse root α of this polynomial; here ord_p denotes the order valuation of $\mathbf{C}_p$, normalized so that $ord_p(p) = 1$. Note also that $ord_p(U)$ is independent of the choice of Φ, since the image of I in GL(U) is a finite group. Clearly, we have $ord_p(U\^(1)) = - ord_p(U) - 1$, so that it is natural to impose the following hypothesis :-

Hypothesis I(p). For each U occurring in the above decomposition, we have $ord_p(U) \neq -1/2$.

For the rest of the paper, we assume that Hypothesis I(p) is valid for our motive M. We then define :-

(a). If $ord_p(U) > -1/2$, then $E_p(\sigma, U, \rho, s) = 1$;

(b). If $ord_p(U) < -1/2$, then $E_p(\sigma, U, \rho, s) = R_p(\sigma, U, \rho, s)$.

Note that the case (a) holds for U if and only if case (b) holds for U^(1), because of Hypothesis I(p). Thus, putting

$$E_p(\sigma, M, \rho, s) = \Pi_{(U)} E_p(\sigma, U, \rho, s),$$

it follows from (48) that the equation (36) is valid, as required.

We now explicitly calculate the the E_p - factors in some simple cases. Let $d_p(\sigma,M)$ denote the number of inverse roots α of the polynomial

$$(\sigma Z_p(M, X))^{-1} = \sigma \det (1 - \Phi. X \mid H_\lambda(M)^I) \tag{49}$$

which satisfy $ord_p(\alpha) < 0$ (by hypothesis, the coefficients of this polynomial are algebraic numbers in $\mathbf{C}$, and we view these as lying in

C_p via the embedding (1)). As usual, we say that M has good reduction at p if the inertia group $I = I_p$ acts trivially on $H_\lambda(M)$ for any prime λ of K not dividing p.

Lemma 7. Assume that M has good reduction at p. Let β (resp. α) run over all inverse roots, counted with multiplicity, of the polynomial (49) such that $\mathrm{ord}_p(\beta) > -1/2$ (resp. $\mathrm{ord}_p(\alpha) < -1/2$). Then

$$E_p(\sigma, M, \rho, s)/L_p(\sigma, M, s) = \Pi_{(\beta)} (1 - \beta p^{-s}).\ \Pi_{(\alpha)} (1 - \alpha^{-1} p^{s-1}).$$

Moreover, if ϕ is a non-trivial character of finite order of G(P/Q), we have

$$E_p(\sigma, M(\phi), \rho, s)/L_p(\sigma, M(\phi), s) = (\delta_\rho(\sigma, \phi)\ c(\phi)^{-s})^{-d_p(\sigma, M)}\ (\Pi_{(\alpha)}\ \alpha)^{-h(\phi)},$$

where $\delta_\rho(\sigma, \phi)$ is the Gauss sum given by (43), and $c(\phi) = p^{h(\phi)}$ is the conductor of ϕ.

Proof. The first assertion is immediate from the definitions because $\varepsilon_p(\sigma, U, \rho, s) = 1$, since U is an unramified representation of W_p. To prove the second, we note that our hypotheses that M has good reduction at p and that p actually divides the conductor of ϕ, imply easily that

$$E_p(\sigma, U(\phi), \rho, s)/L_p(\sigma, U(\phi), s)$$

is equal to 1 or $\varepsilon_p(\sigma, U(\phi), \rho, s)^{-1}$, according as $\mathrm{ord}_p(U)$ is $> -1/2$ or $< -1/2$. Let U be such that $\mathrm{ord}_p(U) < -1/2$. As U is an unramified representation of the Weil group, a standard formula (see (3.4.6) on p.15 of [14]) shows that

$$\varepsilon_p(\sigma, U(\phi), \rho, s) = \varepsilon_p(\sigma, \phi, \rho, s)^{\dim(U)}.\ (\det U)(\Phi^{h(\phi)})\ .$$

But $(\det U)(\Phi)$ = the product of the inverse roots of $\det(1 - \Phi X \mid U)$, and it is well known and readily verified that

$$\varepsilon_p(\sigma, \phi, \rho, s) = \delta_\rho(\sigma, \phi).\ c(\phi)^{-s}.$$

The second assertion now follows.

7. p - adic L-functions. We now take N to be any fixed motive over Q with coefficients in Q itself. Subsequently, we shall take the motive M considered in the earlier sections to be the extension of scalars of N to a variable finite extension K of Q. Our aim is to propose a definition of the p-adic L-function of N. While it is very probable that such p-adic analogues exist for all primes p, we can only make precise conjectures at present when N has good *ordinary* reduction at p. The definition of good reduction at p is given at the end of the previous section. The ordinarity hypothesis is the following condition on the p-adic realisation $V = H_p(N)$ of N as a representation for the local Galois group G_p of the algebraic closure of Q_p over Q_p. There exists a decreasing filtration F^mV of V (with $F^mV = V$ (resp. 0) for m sufficiently small (resp. large)) of Q_p-subspaces, which are stable under the action of G_p, such that, for all m in Z, G_p acts on $F^mV/F^{m+1}V$ via ψ^m; here ψ is the p-adic cyclotomic character (3). We suppose henceforth that N has good ordinary reduction at p. The same is then easily seen to be true for the motive N^(1).

We shall require two additional hypotheses, which are known in many cases, but which we must impose as axioms because of our naive definition of motives. It is well known (see, for example, [9], §6) that our hypothesis that V is ordinary at p implies that it is of Hodge-Tate type. We recall that this means the following. For each n in Z, let $C_p(n)$ be the 1-dimensional vector space over C_p, on which G_p acts via the normal action twisted by ψ^n. Then there is an isomorphism of G_p-modules

$$V \otimes C_p \cong \oplus_{(i \in Z)} C_p(-i)^{h(i)}\ , \text{ where } h(i) = \dim F^iV/F^{i+1}V; \qquad (50)$$

here the tensor product on the left is over Q_p, and where G_p acts on this tensor product in the natural fashion, i.e. $\sigma(u \otimes v) = \sigma(u) \otimes \sigma(v)$ for all σ in G_p. The fist condition we impose is that the integers h(i) appearing in (50) are related to the complex Hodge numbers by

$$h(i) = h(i, w(N) - i) \text{ for all } i \text{ in } Z, \qquad (51)$$

where we recall that w(N) is the weight of the motive N. In fact, (51) has been proven by Faltings [7] when N is of the form $H^k(X)(n)$, for a smooth projective variety X over $\mathbf{Q}$. As before, let

$$Z_p(N, X)^{-1} = \det(1 - \Phi . X \mid H_l(N)),$$

where l is any prime distinct from p. Let d = d(N), and let $\alpha_1, \ldots, \alpha_d$ be the inverse roots in $\mathbf{C}_p$ of this polynomial, taken with multiplicity. Our second assumption is that, for each i in $\mathbf{Z}$, the number of these inverse roots, counted with multiplicity, which satisfy $\mathrm{ord}_p(\alpha) = i$ is equal to the complex Hodge number h(i, w(N) - i). Note, in particular, that this implies that, in the ordinary case, the p-adic order of these inverse roots is an integer, and so Hypothesis I(p) is automatically valid. I understand that the second assumption is known to be true, by the work of Fontaine and Messing (see [8]), when N has good ordinary reduction and is of the form $H^k(X)(n)$, for a smooth projective variety X over $\mathbf{Q}$. We assume from now on that these additional hypotheses are valid. This implies their validity for N^(1).

Lemma 8. Assume that N is critical at s = 0. Let α run over the inverse roots of the polynomial $Z_p(N, X)^{-1}$. The number of these α, counted with multiplicity, satisfying $\mathrm{ord}_p(\alpha) < 0$ is equal to $d^+(N)$. In other words, we have $d_p(N) = d^+(N)$.

Proof. This is plain from (33) and the second additional assumption made above.

Recall that X is the group of all continuous homomorphisms from the Galois group J = G(H/Q) to $\mathbf{C}_p^*$. We write X_{alg} for the subgroup of X consisting of all ξ of the form (11). For such a ξ, we take K to be the finite extension of $\mathbf{Q}$ generated by the values of χ, and let M be the motive over $\mathbf{Q}$, which is given by extending the scalars of N to K in the obvious fashion. We then define

$$N(\xi) = M(n)(\chi). \tag{52}$$

Note that our fixed choice of the embedding (1), together with the fact that we take A to be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, implies that K is

actually given with a canonical embedding $\iota : K \to \mathbf{C}$. In the following, it is convenient and harmless to systematically omit the embedding ι from the notation, and simply write $\Lambda(N(\xi), s)$ instead of $\Lambda(\iota, N(\xi), s)$, etc.

We next consider a question which is important for the study the poles of both complex and p-adic L-functions. Let $Y_p(N)$ be the subspace of $H_p(N)$ given by

$$Y_p(N) = H_p(N)^{G(A/H)} . \tag{53}$$

Since G(A/H) is a normal subgroup of G, it is plain that $Y_p(N)$ is stable under the action of G, and so provides an abelian p-adic representation of G.

Lemma 9. Endowing $Y_p(N) \otimes \mathbf{C}_p$ with the linear action of G (i.e. $\sigma(u \otimes b) = \sigma(u) \otimes b$ for σ in G), it breaks up as a direct sum of G-modules

$$Y_p(N) \otimes \mathbf{C}_p \;\cong\; \oplus_{(\xi \in B(N))}\, \xi^{e(\xi)} \;, \tag{54}$$

where B(N) is some finite subset of X_{alg}, and the $e(\xi)$ are integers ≥ 1. Moreover, each $\xi \in B(N)$ is of the form $\xi = \psi^n \chi$, where n = - w(M)/2 and χ is a character of finite order of G(P/Q).

Proof. Since the representation factors through the Galois group J, and the decomposition group of p in J is equal to J, it suffices to establish (54) as an isomorphism of G_p - modules. Now, viewed as a G_p - module, $Y_p(N)$ is of Hodge-Tate type, because it is easily seen that a sub-representation of a Hodge-Tate representation is again Hodge-Tate. Thus $Y_p(N)$ is an abelian p-adic Hodge-Tate representation of G_p. By a theorem of Tate ([11], §III - 7), this implies that $Y_p(N)$ is locally algebraic (note that in [11], it is necessary to assume that the restriction of the representation to the inertia group is semisimple, but it is pointed out in [12], §2 that this condition is automatically true). Since p is totally ramified in the fixed field of the kernel of the representation on $Y_p(N)$, it follows that, as a G_p-module, $Y_p(N)$ is a direct sum of simple locally algebraic abelian representations. Extending scalars to $\mathbf{C}_p$, we conclude that it is a direct sum of locally algebraic characters of G_p, which factor through J. But such characters are precisely the elements of X_{alg}, and so (54) follows. The final assertion is a consequence of the fact that, for any good prime $q \neq p$, the reciprocal complex roots of $\det(1 - \mathrm{Frob}_q^{-1}.X \mid Y_p(N))$ must have complex absolute value equal to $q^{w(N)/2}$, because of our hypothesis that N has weight w(N).

It is conjectured that the integers $e(\xi)$ occurring in the decomposition (54) are related to the poles of the complex L-functions by

$$e(\xi) = \text{order of pole of } L(M(\xi^{-1}), s) \text{ at } s = 1. \qquad (55)$$

Moreover, for $\xi \in B(N)$, the function $L(M(\xi^{-1}), s)$ should be holomorphic at all points $s \neq 1$, and, for ξ in X_{alg} but not in B(N), this function should be entire.

Assume from now on that our motive N is critical at s=0. We then consider variable twists of N of the form $N(\xi)$, with ξ in X_{alg}, which are also critical at $s = 0$ (infinitely many such ξ clearly exist, since we can, in particular, take ξ to be any character of finite order of J). Let $c^+(N)$ be the Deligne period of N (it is well defined up to multiplication by a non-zero element of Q). As earlier, let $r(N) = \Sigma_{(j<0)} jh(j,k)$. We assume the strong form of the Period Conjecture which is explained in §5. As always, let ρ denote i or -i.

Conjecture A (p-adic version). Assume that N is critical at $s = 0$, and let p be a good ordinary prime for N. For each choice of the Deligne period $c^+(N)$, there exists a unique pseudo-measure $\mu(c^+(N))$ on J as follows: for all ξ in X_{alg} such that (i) $N(\xi)$ is also critical at $s = 0$, and (ii) ξ^{-1} does not belong to B(N) and ξ does not belong to B(N^(1)), we have

$$\int_J \xi \, d\mu(c^+(N)) = \Lambda_{(S)}(N(\xi), \rho, 0)/(c^+(N)(2\pi\rho)^{r(N)}), \qquad (56)$$

where $S = \{\infty, p\}$, and $\Lambda_{(S)}(N(\xi), \rho, s)$ is the modified L-function defined in §5, for the standard embedding $\iota : K \to C$.

Remarks.

1. Using Lemmas 5, 7, and 8, we see easily that the right hand side of (56) is independent of the choice of ρ, and hence so is the pseudo-measure $\mu(c^+(N))$.

2. Using the Period Conjecture of §5, and taking into account all the embeddings of K in C, one can show that the pseudo-measure $\mu(c^+(N))$ takes values in Q_p.

3. The following conjecture about the possible poles of the pseudo-measure $\mu(c^+(N))$ should replace that proposed in our earlier papers [3], [4]. Our previous conjecture was too strong because it failed to take into account possible twisting by the characters of finite order of the Galois group J.

Holomorphy Conjecture (p-adic version). Let B(N) and B(N^(1)) be the subsets of X_{alg} occurring in the decomposition (54) for N and N^(1), respectively. Then there exists a non-zero b in Z_p such that

$$b \, \Pi_{(\xi \in B(N))} (\xi^{-1}(\sigma(\xi)) - \sigma(\xi))^{e(\xi)} \, \Pi_{(\eta \in B(N^\wedge(1)))} (\eta(\sigma(\eta)) - \sigma(\eta))^{e(\eta)} \mu(c^+(M))$$

belongs to the Iwasawa algebra $\mathfrak{I} = Z_p[[J]]$, for all choices of $\sigma(\xi)$ and $\sigma(\eta)$ in J.

In parallel with (55), it seems reasonable to conjecture the even stronger assertion, that $\mu(c^+(N))$ will have poles of exact order $e(\xi)$ at ξ^{-1} for ξ in B(N), and of exact order $e(\eta)$ at each η in B(N^(1)).

4. If $N = Q(m)$, where m is an odd negative integer, then N is critical at $s = 0$, and it is easily seen that Theorem 1 (with $\phi = 1$) shows that both Conjecture A and the Holomorphy Conjecture above do indeed hold for this motive. For further examples, see [4].

The pseudo-measure $\mu(c^+(N))$ satisfies a simple p-adic analogue of the functional equation of the complex L-function. Recall that $\mu \to \mu^\#$ is the involution of the ring of pseudo-measures on J, which is induced by sending σ in J to σ^{-1}. The conductor of N is an integral ideal of Z, which is prime to p because N has good reduction at p, and we write $\sigma(N)$ for its Artin symbol in J.

Put

$$\gamma(N) =$$

$$\varepsilon(N, 0) \,.\, (2\pi\rho)^{r(N) - r(N^\wedge(1))} (-1)^{r(N^\wedge(1))} \varepsilon_\infty (N, \rho, 0)^{-1} . c^+(N)/c^+(N^\wedge(1)).$$

This number does depend on the choice of the periods $c^+(N)$, and $c^+(N^\wedge(1))$, but the arguments of §5 show that it is independent of the choice of ρ. Moreover, the arguments of [6], §5 prove that it lies in Q.

p - adic functional equation. We have

$$\mu(c^+(N)) = \gamma(N) .\ \mu(c^+(N^\wedge(1))^{\#} \,.\, \sigma(M)^{\#}, \tag{57}$$

where $\sigma(M)$ denotes the Artin symbol of the conductor of M in the Galois group J.

Proof. It suffices to show that, for any ξ in X_{alg} satisfying the conditions set out in Conjecture A above, the integrals of ξ against both sides of (57) are equal. This follows immediately by combining (56), the modified functional equation (37), and the well known formula that, for $q \neq \infty$, p, we have

$$\varepsilon_q(N(\xi), \rho, 0) = \varepsilon_q(N, \rho, 0)\, \xi(\mathrm{Frob}_q^{-1})^{a(q)},$$

where $q^{a(q)}$ is the power of q in the conductor of M; this latter formula is valid because ξ is unramified at q.

In view of (57), we see that $\gamma(N)$ plays the role of a global p-adic ε-factor. We only make one observation here about its properties.

Lemma 10. Assume that w(N) is odd. Then

$$\gamma(N) = \varepsilon(N, 0) .\ (2\pi)^{(1 + w(N))d^+(N)} \,.\, c^+(N)/c^+(N^\wedge(1)). \tag{58}$$

In particular, if $N = N^\wedge(1)$, and if we take $c^+(N) = c^+(N^\wedge(1))$, then

$$\gamma(N) = \varepsilon(N, 0)\,. \tag{59}$$

Proof. We note that (59) follows immediately from (58), since $N = N^\wedge(1)$ implies that $w(N) = -1$. To prove (58), we observe that, because w(N) is odd, the only terms in the Hodge - decomposition (21) are the $H^{i,j}(N)$ with $i \neq j$, and hence the explicit formula for the infinite ε-factors, given in [6], §5, shows that

$$\varepsilon_\infty (N, \rho, 0) = \rho^{b(N)}, \text{ where } b(N) = \Sigma_{(j<0)} (w(N) - 2j + 1)h(j, w(N) - j).$$

In view of (33), we conclude that

$$\varepsilon_\infty (N, \rho, 0) = \rho^{(1 + w(N))d^+(N)} .\ (-1)^{r(N)}\,. \tag{60}$$

On the other hand, because w(N) is odd, it is readily verified that

$$r(N) - r(N^\wedge(1)) = (1+ w(N))d^+(N).$$

Since the right hand side of this last formula is even, (58) now follows from (60) and the definition of $\gamma(N)$.

We conclude by remarking that (59) is exactly what would be predicted by the main conjecture and algebraic arguments involving Iwasawa modules (see Proposition 1 of Greenberg's article in this volume).

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The Beilinson conjectures

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Introduction

The Beilinson conjectures describe the leading coefficients of L-series of varieties over number fields up to rational factors in terms of generalized regulators. We begin with a short but almost selfcontained introduction to this circle of ideas. This is possible by using Bloch's description of Beilinson's motivic cohomology and regulator map in terms of higher Chow groups and generalized cycle maps. Here we follow **[Bl3]** rather closely. We will then sketch how much of the known evidence in favour of these conjectures — to the left of the central point — can be obtained in a uniform way. The basic construction is Beilinson's Eisenstein symbol which will be explained in some detail. Finally in an appendix a map is constructed from higher Chow theory to a suitable Ext-group in the category of mixed motives as defined by Deligne and Jannsen. This smooths the way towards an interpretation of Beilinson's conjectures in terms of a Deligne conjecture for critical mixed motives **[Sc2]**. It also explains how work of Harder **[Ha2]** and Anderson fits into the picture.

For further preliminary reading on the Beilinson conjectures, one should consult the Bourbaki seminar of Soulé **[So1]**, the survey article by Ramakrishnan **[Ra2]** and the introductory article by Schneider **[Sch]**. For the full story see the book **[RSS]** and of course Beilinson's original paper **[Be1]**. Here one will also find the conjectures for the central and near-central points, which for brevity we have omitted here.

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